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A Simple Analytic Formula for the Number of Theoretical Plates in Distillation

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Abstract

The Lewis equation for the number of theoretical plates in a distillation column has been simplified. The simplification is not at the expense of accuracy. On the contrary, it is shown that the simple equation leads almost exclusively to more accurate results.

The number of theoretical plates is obtained graphically by counting the number of steps resulting from drawing horizontal and vertical lines between two lines, one of which is called the operating line and the other the equilibrium line. Either of the two lines may be a straight or a curved line. In some instances it is possible to calculate analytically the number of these steps as in the following cases.

CASE 1

Both equilibrium and operating lines are straight parallel lines as in Fig. 1. The corresponding equations are:

$$y_e = mx_e + \theta_e \tag{1}$$

and

$$y = mx + \theta \tag{2}$$

where m is the slope and θ is the intercept on the y axis.

The number of steps between points x_a and x_b is equal to

$$N = \frac{x_b - x_a}{\text{horizontal width}} \tag{3}$$

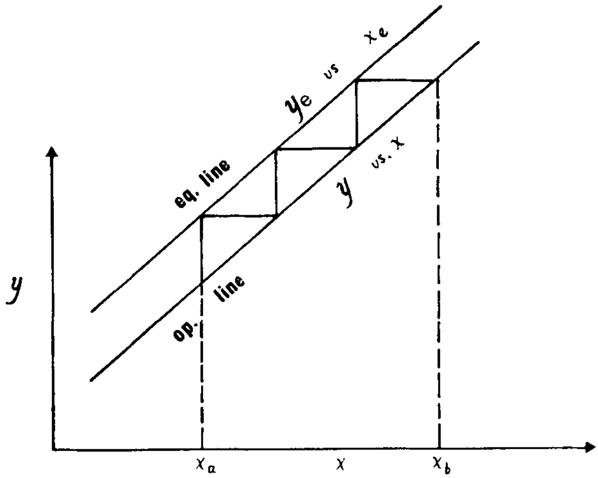

FIG. 1. Graphical determination of theoretical plates (operating and equilibrium lines are parallel).

The horizontal width is obtained by equating y_e and y, resulting in

$$x - x_e = \frac{\theta_e - \theta}{m} \tag{4}$$

and hence

$$N = \frac{m}{\theta_e - \theta}(x_b - x_a) \tag{5}$$

CASE 2

Both operating and equilibrium lines are straight lines having different slopes as in Fig. 2. We have

$$y_e = m_e x_e + \theta_e \tag{1}$$

and

$$y = mx + \theta \tag{6}$$

for the equilibrium and operating lines, respectively.

The relation between x_{n+1} and x_n is obtained by equating y_e and y, and thus

$$x_{n+1} = \frac{m_e}{m} x_n + \frac{\theta_e - \theta}{m} \tag{7}$$

If we denote m_e/m by β and $(\theta_e - \theta)/m$ by γ, then

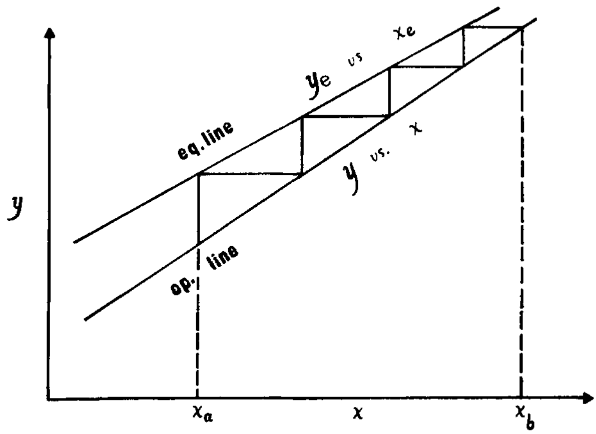

FIG. 2. Graphical determination of theoretical plates (operating and equilibrium lines are not parallel).

$$x_1 = \beta x_0 + \gamma$$
$$x_2 = \beta x_1 + \gamma = \beta^2 x_0 + \beta\gamma + \gamma$$
$$x_N = \beta^N x_0 + \gamma(1 + \beta + \beta^2 + \cdots + \beta^{N-1})$$
$$= \beta^N x_0 + \gamma \frac{\beta^N - 1}{\beta - 1}$$

or

$$N = \frac{1}{\ln \beta} \ln \frac{x_b + \dfrac{\gamma}{\beta - 1}}{x_a + \dfrac{\gamma}{\beta - 1}} \tag{8}$$

CASE 3

The case where the operating and equilibrium lines are given by the equations

$$y = x \tag{9}$$

and

$$y_e = \frac{\alpha x}{1 + (\alpha - 1)x} \tag{10}$$

where α is the relative volatility.

This is the case encountered in the distillation of a binary ideal mixture using total reflux. It leads to the Fenske equation:

$$N = \frac{1}{\ln \alpha} \ln \frac{x_a(1 - x_b)}{x_b(1 - x_a)} \tag{11}$$

THE CASE OF FINITE REFLUX

In the case of a binary ideal mixture using a finite reflux rlatio, the analytic solution becomes complex and a knowledge of the calculus of finite differences is necessary.

An approximate expression was deduced by Lewis (*1*) starting from the differential equation

$$x_{n+1} = x_n + \frac{dx}{dn} \tag{12}$$

where x_n is the x coordinate of step n, leading to

$$N = \int dn = \int_{x_a}^{x_b} \frac{dx}{x_{n+1} - x_n} \tag{13}$$

Here, the operating line is given by the equation

$$y = mx + \theta \tag{6}$$

and the equilibrium line is given by Eq. (10). From Eqs. (6) and (10), one obtains

$$\frac{\alpha x_n}{1 + (\alpha - 1)x_n} = y_{n+1} = mx_{n+1} + \theta$$

and hence

$$x_{n+1} = \frac{1}{m}\left[\frac{\alpha x_n}{1 + (\alpha - 1)x_n} - \theta\right]$$

substituting in Eq. (13) and dropping the subscript n gives

$$N_I = \int_{x_a}^{x_b} \frac{dx}{\frac{1}{m}\left[\frac{\alpha x}{1 + (\alpha - 1)x} - \theta\right] - x} \tag{14}$$

This is a standard integral which leads to the Lewis equation (*2, 3*)

$$N_I = \frac{b/2 - m}{\sqrt{q}}\left[\ln \frac{x + A}{x + B}\right]_{x_a}^{x_b} - \frac{1}{2}[\ln (ax^2 + bx + c)]_{x_a}^{x_b} \tag{15}$$

where

$$a = m(\alpha - 1)$$

$$b = \theta(\alpha - 1) + m - \alpha$$

$$c = \theta$$
$$q = b^2 - 4ac$$
$$A = \frac{b - \sqrt{q}}{2a}$$
$$B = \frac{b + \sqrt{q}}{2a}$$

The second term on the right-hand side of Eq. (15) is not necessary. In fact, as will be shown below, its omission leads to an equation which is superior both in simplicity and accuracy.

For this, we introduce an alternative to the Lewis differential equation, namely:

$$x_n = x_{n-1} + \frac{dx}{dn} \tag{16}$$

leading to

$$N_{\text{II}} = \int_{x_a}^{x_b} \frac{dx}{x_n - x_{n-1}} \tag{17}$$

Again, from Eqs. (6) and (10) one gets

$$mx_n + \theta = y_{n-1} = \frac{\alpha x_{n-1}}{1 + (\alpha - 1)x_{n-1}} \tag{18}$$

or

$$x_{n-1} = \frac{mx_n + \theta}{\alpha - (mx_n + \theta)(\alpha - 1)} \tag{19}$$

Substituting for x_{n-1} in Eq. (17) and dropping the subscript n, one gets

$$N_{\text{II}} = \int_{x_a}^{x_b} \frac{dx}{x - \dfrac{mx + \theta}{\alpha - (mx + \theta)(\alpha - 1)}} \tag{20}$$

This is also a standard integral which leads to

$$N_{\text{II}} = \frac{b/2 - m}{\sqrt{q}}\left[\ln \frac{x + A}{x + B}\right]_{x_a}^{x_b} + \frac{1}{2}[\ln (ax^2 + bx + c)]_{x_a}^{x_b} \tag{21}$$

and

$$N' = \frac{N_{\text{I}} + N_{\text{II}}}{2} = \frac{b/2 - m}{\sqrt{q}}\left[\ln \frac{x + A}{x + B}\right]_{x_a}^{x_b} \tag{22}$$

Equation (22) may also be obtained directly and more easily by adding integrals (14) and (20).

The number of plates N' calculated from Eq. (22) is more accurate than that calculated from Eqs. (15) or (21). One can prove this fact analytically for Cases 2 and 3. For the case of finite reflux ratio, a number of examples were chosen at random and in each case, $N_{\rm I}$, $N_{\rm II}$, and N' were calculated and compared with the exact value N calculated from the exact recursion formula. It was found that N' was almost exclusively a better approximation than either $N_{\rm I}$ or $N_{\rm II}$.

Applying Eqs. (13) and (17) to Case 2 discussed above, one finds on substituting from Eq. (7) into Eq. (13) and integrating between limits x_a and x_b that

$$N_{\rm I} = \frac{1}{\beta - 1} \ln \frac{x_b + \dfrac{\gamma}{\beta - 1}}{x_a + \dfrac{\gamma}{\beta - 1}} \tag{23}$$

Similarly, on equating y_e and y in Eqs. (1) and (6), one gets

$$x_{n-1} = \frac{m}{m_e} x_n + \frac{\theta - \theta_e}{m_e} \tag{24}$$

Substituting in Eq. (17), dropping the subscript n, and integrating gives

$$N_{\rm II} = \frac{\beta}{\beta - 1} \ln \frac{x_b + \dfrac{\gamma}{\beta - 1}}{x_a + \dfrac{\gamma}{\beta - 1}} \tag{25}$$

and

$$N' = \frac{N_{\rm I} + N_{\rm II}}{2} = \frac{\beta + 1}{2(\beta - 1)} \ln \frac{x_b + \dfrac{\gamma}{\beta - 1}}{x_a + \dfrac{\gamma}{\beta - 1}} \tag{26}$$

In all cases, Eq. (26) leads to much more accurate values than do Eqs. (23) and (25), as shown in Table 1.

On applying Eqs. (13) and (17) to Case 3, one gets

$$N_{\rm I} = \frac{1}{\alpha - 1} \ln \frac{x_D(1 - x_W)}{x_W(1 - x_D)} - \ln \frac{1 - x_D}{1 - x_W} \tag{27}$$

$$N_{\rm II} = \frac{\alpha}{\alpha - 1} \ln \frac{x_D(1 - x_W)}{x_W(1 - x_D)} + \ln \frac{1 - x_D}{1 - x_W} \tag{28}$$

$$N' = \frac{N_{\rm I} + N_{\rm II}}{2} = \frac{\alpha + 1}{2(\alpha - 1)} \ln \frac{x_D(1 - x_W)}{x_W(1 - x_D)} \tag{29}$$

D and W refer to distillate and bottoms, respectively.

TABLE 1

Comparison between Different Relative N Values for Case 2. Also Shown Is the Superiority of N' Values over N_I and N_{II}

β	N_I/N, $\frac{\ln \beta}{\beta - 1}$	N_{II}/N, $\frac{\beta \ln \beta}{\beta - 1}$	N'/N, $\frac{(\beta + 1) \ln \beta}{2(\beta - 1)}$
0.8	1.116	0.893	1.004
0.9	1.054	0.948	1.001
1	1	1	1
1.1	0.953	1.048	1.001
1.2	0.912	1.094	1.003

Here also, Eq. (29) leads almost exclusively to better answers than those obtained from Eqs. (27) and (28).

The ratio N'/N is given by the same formula deduced for N'/N for Case 2 except that the range of α values may differ from that of β values—as shown in Table 2.

The following example illustrates the application of Eqs. (15), (21), and (22). The data in it is taken from an example in Ref. *2*, p. 122.

Example. It is required to calculate N, N_I, N_{II}, and N' when a 50–50 mol-% benzene–toluene mixture is to be separated in a distillation column into two fractions: A top product containing 95 mol-% benzene and a bottom product containing 95 mol-% toluene, reflux ratio $R = 3$, relative volatility $\alpha = 2.5$, and the feed introduced as a saturated liquid.

Basis. 10 mols feed:

$D = 5$ mols, $W = 5$ mols

above feed plate: $L = 15$ mols, $V = 20$ mols

below feed plate: $L = 25$ mols, $V = 20$ mols

L and V refer to liquid and vapor, respectively.

TABLE 2

Values of N'/N for Different Values of α for Case 3

α	N'/N $\frac{(\alpha + 1) \ln \alpha}{2(\alpha - 1)}$
1	1
1.1	1.001
1.2	1.003
1.5	1.014
2	1.040
2.5	1.069
3	1.099

Upper operating line.

$$y = mx + \theta$$

$$m = L/V = \frac{15}{20} = 0.75$$

$$\theta = x_D \frac{D}{V} = 0.95 \times \frac{5}{20} = 0.2375$$

Hence

$$y = 0.75x + 0.2375$$

Lower operating line.

$$y = mx + \theta$$

$$m = L/V = \frac{25}{20} = 1.25$$

$$\theta = -x_w \frac{W}{V} = -0.05 \times \frac{5}{20} = -0.0125$$

hence

$$y = 1.25x - 0.0125$$

Equilibrium curve.

$$y = \frac{\alpha x}{1 + (\alpha - 1)x}$$

Recursion formula. Knowing x_n, then

$$y_{n+1} = mx_n + \theta \tag{30}$$

but

$$y_{n+1} = \frac{\alpha x_{n+1}}{1 + (\alpha - 1)x_{n+1}} \tag{31}$$

and hence

$$x_{n+1} = \frac{mx_n + \theta}{\alpha - (\alpha - 1)(mx_n + \theta)} \tag{32}$$

For the upper operating line, $m = 0.75$, $\theta = 0.2375$, and $\alpha = 2.5$:

$$x_{n+1} = \frac{0.75x_n + 0.2375}{2.5 - 1.5(0.75x_n + 0.2375)} \tag{33}$$

For the lower operating line, $m = 1.25$, $\theta = -0.0125$, and $\alpha = 2.5$:

$$x_{n+1} = \frac{1.25x_n - 0.0125}{2.5 - 1.5(1.25x_n - 0.0125)} \tag{34}$$

Start from $x_0 = 0.95$ and apply Eq. (33) until x_n is less than 0.5, after which Eq. (34) is applied. The results are tabulated in Table 3.

Using linear interpolation:

$$N_{\text{total}} = 8 + \frac{0.03811}{0.04662} = 8.82 \text{ theoretical plates}$$

$$N_{\text{top}} = 4 + \frac{0.01635}{0.11660} = 4.14 \text{ theoretical plates}$$

$$N_{\text{total}} = N_{\text{top}} + N_{\text{bottom}}$$

$$N_{\text{bottom}} = 8.82 - 4.14 = 4.68 \text{ theoretical plates}$$

N_{I}, N_{II}, and N' were calculated from Eqs. (15), (21), and (22), respectively, both above and below the feed plate, and the results are tabulated in Table 4.

Table 4 shows the better accuracy of Eq. (22) compared to the more complex Eqs. (15) and (21).

The difference between N' and N is always a small fraction of one

TABLE 3

List of Exact x_n Values Obtained from Recursion Formulas (33) and (34)

Number	x_n	Number	x_n
x_0	0.95	x_5	0.39975
x_1	0.88372	x_6	0.27537
x_2	0.78316	x_7	0.16565
x_3	0.65326	x_8	0.08811
x_4	0.51635	x_9	0.04149

TABLE 4

List of Different N Values Calculated from Data in the Example, and Their Deviations from the Exact Value

	Upper section		Lower section	
	Theoretical plates	Error	Theoretical plates	Error
N	4.14	—	4.68	—
N_{I}	4.74	+0.60	4.41	−0.26
N_{II}	3.82	−0.32	5.33	+0.65
N'	4.28	+0.14	4.87	+0.19

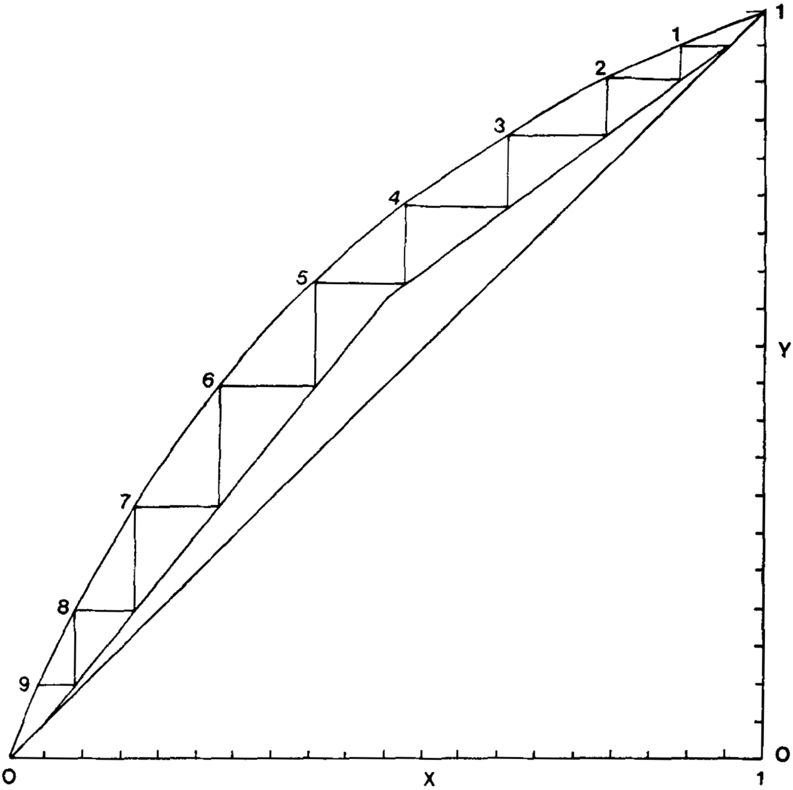

FIG. 3. Graphical determination of the number of theoretical plates for the benzene–toluene example.

theoretical plate, regardless of the value of N. $N' = N + \delta$, where δ is a small positive fraction which is very seldom greater than 0.2.

Figure 3 gives the graphical solution to this example.

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